

**Azumaya structure on D-branes
and deformations and resolutions of a conifold revisited:
Klebanov-Strassler-Witten vs. Polchinski-Grothendieck**

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Abstract

In this sequel to [L-Y1], [L-L-S-Y], and [L-Y2] (respectively arXiv:0709.1515 [math.AG], arXiv:0809.2121 [math.AG], and arXiv:0901.0342 [math.AG]), we study a D-brane probe on a conifold from the viewpoint of the Azumaya structure on D-branes and toric geometry. The details of how deformations and resolutions of the standard toric conifold Y can be obtained via morphisms from Azumaya points are given. This should be compared with the quantum-field-theoretic/D-brany picture of deformations and resolutions of a conifold via a D-brane probe sitting at the conifold singularity in the work of Klebanov and Witten [K-W] (arXiv:hep-th/9807080) and Klebanov and Strasser [K-S] (arXiv:hep-th/0007191). A comparison with resolutions via noncommutative desingularizations is given in the end.

Key words: D-brane, Azumaya structure, Polchinski-Grothendieck Ansatz, Azumaya point, conifold.

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*In memory of a young string theorist Ti-Ming Chiang,
whose path I crossed accidentally and so briefly.[†]*

[†]*From C.-H.L.* During the years I was attending Prof. Candelas's group meetings, I learned more about Calabi-Yau manifolds and mirror symmetry and got very fascinated by the works from Brian Greene's group. Because of this, I felt particularly lucky knowing later that I was going to meet one of his students, Ti-Ming, - a young string theorist with a PhD from Cornell at his very early 20's - and perhaps to cooperate with him. Unfortunately that anticipated cooperation never happened. Ti-Ming had become unwell just before I resettled. Except the visits to him at the hospital and some chats when he showed up in the office, I didn't really get the opportunity to interact with him intellectually. Further afterwards I was informed of Ti-Ming's passing away. Like a shooting star he reveals his shining so briefly and then disappears. The current work is the last piece of Part 1 of the D-brane project. It is grouped with the earlier D(1), D(2), D(3) under the hidden collective title: "Azumaya structure on D-branes and its tests". Here we address in particular a conifold from the viewpoint of a D-brane probe with an Azumaya structure. This is a theme Ti-Ming may have felt interested in as well, should he still work on string theory, since conifolds have play a role in understanding the duality web of Calabi-Yau threefolds - a theme Ti-Ming once worked on - and D-brane resolution of singularities is a theme Brian Greene's group once pursued vigorously. We thus dedicate this work to the memory of Ti-Ming.

0. Introduction and outline.

Conifolds, i.e. Calabi-Yau threefolds with ordinary double-points, have been playing special roles at various stages of string theory.¹ In this sequel to [L-Y1], [L-L-S-Y], and [L-Y2], we study a D-brane probe on a conifold from the viewpoint of Azumaya structure on D-branes and toric geometry. This should be compared with the quantum-field-theoretic/D-brany picture of deformations and resolutions of a conifold in the work of Klebanov and Strasser [K-S] and Klebanov and Witten [K-W].

Effective-space-time-filling D3-brane at a conifold singularity.

In [K-W], Klebanov and Witten studied the $d = 4$, $N = 1$ superconformal field theory (SCFT)² on the D3-brane world-volume X ($\simeq \mathbb{R}^4$ topologically) that is embedded in the product space-time $\mathbb{M}^{3+1} \times Y$ as $\mathbb{M}^{3+1} \times \{\mathbf{0}\}$,³ and its supergravity dual - a compactification of $d = 10$, type-IIB supergravity theory on $\text{AdS}^5 \times (S^3 \times S^2)$ - along the line of the AdS/CFT correspondence of Maldacena [Ma]. Here \mathbb{M}^{3+1} is the $d = 3 + 1$ Minkowski space-time, Y is the conifold $\{z_1 z_2 - z_3 z_4 = 0\} \subset \mathbb{C}^4$ (with coordinates (z_1, z_2, z_3, z_4)), $\mathbf{0}$ is the conifold singularity on Y , and AdS^5 is the $d = 4 + 1$ anti-de Sitter space-time.

In the simplest case when there is a single D3-brane sitting at the conifold point of Y , the *classical moduli space* of the *supersymmetric vacua* of the associated $U(1)$ super-Yang-Mills theory coupled with matter on the D3-brane world-volume comes from the D -term of the *vector multiplet* and the coefficient $\zeta \in \mathbb{R}$ of the *Fayet-Iliopoulos term* in the Lagrangian.⁴ By varying ζ , one realizes the two small resolutions, Y_+ and Y_- , of Y as the classical moduli space Y_ζ of the above $d = 4$ SCFT.⁵ A flop $X_+ \dashrightarrow Y_-$ happens when Y_ζ crosses over $\zeta = 0$.

To describe the physics for N -many parallel D3-branes sitting at the conifold singularity, Klebanov and Witten proposed to enlarge the gauge group for the super-Yang-Mills theory on the common world-volume of the stacked D3-brane to $U(N) \times U(N)$ (rather than the naive $U(N)$) and introduce a *superpotential* W for the chiral multiplets. The classical moduli space of the theory comes from a system with equations of the type above (i.e. D -term equations) and equations from the superpotential term W (i.e. F -term equations). In particular, the N -fold symmetric product $\text{Sym}^N Y$ of Y can be realized as the classical moduli space of the $d = 4$ SCFT on the D3-brane world-volume with $\zeta = 0$.

In [K-S], Klebanov-Strassler studied further $d = 4$, $N = 1$ supersymmetric quantum field theory (SQFT) on the D3-brane world-volume that arises from a D3-brane configuration with both N -many above *full*/free D3-branes and M -many new *fractional*/trapped D3-branes⁶ sitting

¹Readers are referred to, for example, [C-dIO] (1989); [Stro], [G-M-S], [C-G-G-K] (1995); [G-V] (1998); [Be], [C-F-I-K-V] (2001) and references therein to get a glimpse of conifolds in string theory around the decade 1990s.

²There will be a few standard physicists' conventional notations in this highlight of the relevant part of [K-W] and [K-S]: N that counts the *number of supersymmetries* (susy) via the multiple number of minimal susy numbers in each space-time dimension vs. N that appears in the *gauge group* $U(N)$ or $SU(N)$ vs. N that counts the *multiplicity* of stacked D-branes.

³In string-theorist's terminology, the D3-brane is "*sitting at the conifold singularity*". We will also adopt this phrasing for convenience. Note that in such a setting, the internal part is a D0-brane on the conifold Y . The latter is what we will study in this work.

⁴ ζ is part of the parameters to give local coordinates of the *Wilson's theory-space* in the problem; cf. [L-Y2: Introduction] for brief words. See also [W-B] for the standard SUSY jargon.

⁵See also [Wi] and [D-M] for details of such a construction.

⁶See [G-K] and references therein for the detail of such *fractional D-branes*.

at the conifold singularity $\mathbf{0}$ of Y . For infrared physics, the theory now has the gauge group $SU(N + M) \times SU(N)$. It follows from the work of Affleck, Dine, and Seiberg [A-D-S]⁷ that an *additional term* to the previous superpotential W is now *dynamically generated*. This deforms the classical moduli space of SUSY vacua of the $d = 4$ SQFT on the D3-brane world-volume. In the simplest case when $N = M = 1$, this enforces a deformation of the classical moduli space from a conifold to a deformed conifold $Y' (\simeq T^*S^3$ topologically). Cf. FIGURE 0-1.

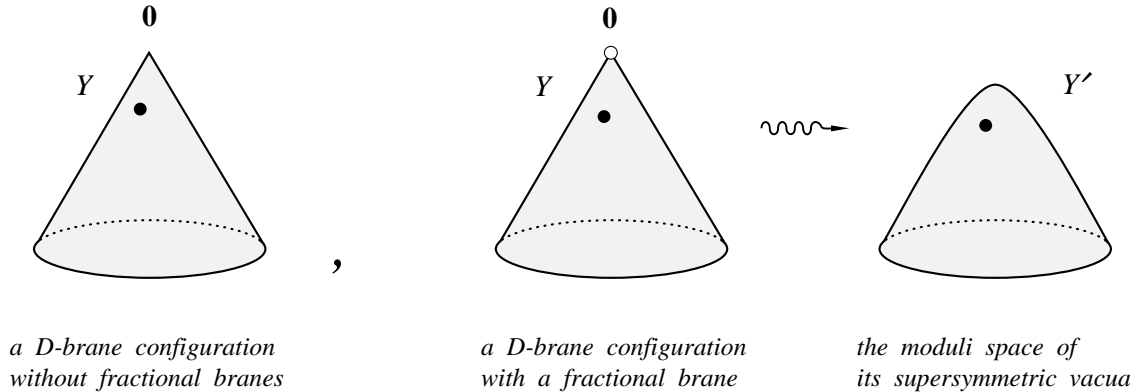


FIGURE 0-1. (Cf. [Stra: Figures 25, 26, 27].) When a fractional/trapped D3-brane sits at the conifold singularity $\mathbf{0} \in Y$, the full/free D3-brane “sees” a smooth deformed conifold $Y' (\simeq T^*S^3$ topologically) as its classical vacua manifold. I.e., in very low energy for this situation the free D3-brane “feels” as if it lives on Y' instead of Y ! In the figure, a full D3-brane is indicated by \bullet while a fractional D3-brane by \circ .

While giving only a highlight of key points in [K-W] and [K-S] that are most relevant to us, we should remark that, in addition to further quantum-field-theoretical issues on the gauge theory side, there is also a gravity side of the story that was studied in [K-W] and [K-S].⁸

Azumaya structure on D-branes and its tests.

In D(1) [L-Y1], D(2) [L-L-S-Y], D(3) [L-Y2] and the current work D(4), we illuminate the Azumaya geometry as a key feature of the geometry on D-brane world-volumes in the algebro-geometric category. These four together center around the very remark of Polchinski:

([Po: vol. 1, Sec. 8.7, p. 272]) “For the collective coordinate X^μ , however, the meaning is mysterious: the collective coordinates for the embedding of n D-branes in space-time are now enlarged to $n \times n$ matrices. This ‘noncommutative geometry’ has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important hint about the nature of space-time.”,

⁷See also [Arg: Chapter 3] and [Te: Chapter 9].

⁸See [A-G-M-O-O] and [Stra] for a review with more emphasis on respectively the gravity and the gauge theory side in the correspondence; e.g. [G-K], [K-N] for developments between [K-W] and [K-S]; and e.g. [D-K-S] for a more recent study.

which was taken as a guiding question as to what a D-brane is in this project, cf. [L-Y1: Sec. 2.2]. D(2), D(3), and the current D(4) are meant to give more explanations of the highlight [L-Y1: Sec. 4.5]. In this consecutive series of four, we learned that :

Lesson 0.1 [Azumaya structure on D-branes]. *This “enhancement to $n \times n$ matrices” Polchinski alluded to says even more fundamentally the nature of D-branes themselves, i.e. the Azumaya structure thereupon. This structure gives them the power to detect the nature of space-time. We also learned that Azumaya structures on D-branes and morphisms therefrom can be used to reproduce/explain several stringy/brany phenomena of stringy or quantum-field-theoretical origin that are very surprising/mysterious at a first mathematical glance.*

This is a basic test to ourselves to believe that Azumaya structures play a special role in understanding/describing D-branes in string theory. Having said this, we should however mention that D-brane remains a very complicated object and the Azumaya structure addressed here is only a part of it. Further issues are investigated in separate works.

Convention. Standard notations, terminology, operations, facts in (1) physics aspects of strings and D-branes; (2) algebraic geometry; (3) toric geometry can be found respectively in (1) [Po], [Jo]; (2) [Ha]; (3) [Fu].

- *Noncommutative algebraic geometry* is a very technical topic. For the current work, [Art] of Artin, [K-R] of Kontsevich and Rosenberg, and [leB1] of Le Bruyn are particularly relevant. See [L-Y1: References] for more references.

Outline.

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1 D-branes in an affine noncommutative space.

We recall definitions and notions in [L-Y1] that are needed for the current work. Readers are referred to *ibidem* for more details and references. See also [L-L-S-Y] and [L-Y2] for further explanations and examples.

Affine noncommutative spaces and their morphisms.

An *affine noncommutative space* over \mathbb{C} is meant to be a “space” $Space R$ that is associated to an associative unital \mathbb{C} -algebra R . In general, it can be tricky to truly realize $Space R$ as a set of points with a topology in a natural/functorial way. However, “geometric” notions can still be pursued - despite not knowing what $Space R$ really is - via imposing the fundamental geometry/algebra ansatz:

- [*geometry = algebra*] The correspondence $R \leftrightarrow Space R$ gives a contravariant equivalence between the category $Alg_{\mathbb{C}}$ of associative unital \mathbb{C} -algebras and the category $AffineSpace_{\mathbb{C}}$ of “affine noncommutative spaces” over \mathbb{C} .

For example,

Definition 1.1. [smooth affine noncommutative space]. ([C-Q: Sec. 3], [K-R: Sec. 1.1.4].)

An affine noncommutative space $Space R$ over \mathbb{C} is said to be *smooth* if the associative unital \mathbb{C} -algebra R is finitely generated and satisfies the following property:

- (*lifting property for nilpotent extensions*) for any \mathbb{C} -algebra S , two-sided nilpotent ideal $I \subset R$ (i.e. $I = BIB$ and $I^n = 0$ for $n \gg 0$), and \mathbb{C} -algebra homomorphism $h : R \rightarrow B/I$, there exists an \mathbb{C} -algebra homomorphism $\tilde{h} : R \rightarrow B$ such that the diagram

$$\begin{array}{ccc} & & B \\ & \nearrow \tilde{h} & \downarrow \\ R & \xrightarrow{h} & B/I \end{array}$$

commutes. Here $B \rightarrow B/I$ is the quotient map.

The following two classes of smooth affine noncommutative spaces are used in this work.

Example 1.2. [noncommutative affine space]. ([K-R: Sec. 2: Example (E1)].) The *noncommutative affine n -space* $N\mathbb{A}^n := Space(\mathbb{C}\langle \xi_1, \dots, \xi_n \rangle)$ over \mathbb{C} is smooth. Here $\mathbb{C}\langle \xi_1, \dots, \xi_n \rangle$ is the associative unital \mathbb{C} -algebra freely generated by the elements in the set $\{\xi_1, \dots, \xi_n\}$.

Example 1.3. [Azumaya-type noncommutative space]. ([C-Q: Sec. 5 and Proposition 6.2], [K-R: Sec. 1.2, Examples (E2) and (C4)].) Let $M_r(R)$ be the \mathbb{C} -algebra of $r \times r$ -matrices with entries in a commutative regular \mathbb{C} -algebra R . Then the *Azumaya-type noncommutative space* $Space M_r(R)$ is smooth (over \mathbb{C}). Furthermore, it is also smooth over $Spec R$.

As a consequence of the Geometry/Algebra Ansatz, a *morphism* $\varphi : X = Space R \rightarrow Y = Space S$ is defined contravariantly to be a \mathbb{C} -algebra homomorphism $\varphi^\# : S \rightarrow R$. The *image*, denoted $Im \varphi$ or $\varphi(X)$, of X under φ is defined to be $Space(S/Ker \varphi^\#)$. The latter is canonically included in Y via the morphism $\iota : \varphi(X) \hookrightarrow Y$ defined by the \mathbb{C} -algebra quotient-homomorphism $\iota^\# : S \rightarrow S/Ker \varphi^\#$. This extends what is done in Grothendieck’s theory of (commutative) schemes. The benefit of thinking a morphism between affine noncommutative spaces this way is actually *two folds*:

(1) *As a functor of point*: The space $X = \text{Space } R$ defines a functor

$$\begin{aligned} h_X &: \text{AffineSpace}_{\mathbb{C}} \longrightarrow \text{Set}^{\circ} \\ Y &\longmapsto \text{Mor}(Y, X); \end{aligned}$$

i.e. a functor

$$\begin{aligned} h_R &: \text{Alg}_{\mathbb{C}} \longrightarrow \text{Set} \\ S &\longmapsto \text{Hom}(R, S). \end{aligned}$$

Here Set is the category of sets, Set° its opposite category, and $\text{Hom}(R, S)$ is the set of \mathbb{C} -algebra-homomorphisms.

(2) *As a probe*: $X = \text{Space } R$ defines another functor

$$\begin{aligned} g_X &: \text{AffineSpace}_{\mathbb{C}} \longrightarrow \text{Set} \\ Y &\longmapsto \text{Mor}(X, Y); \end{aligned}$$

i.e. a functor

$$\begin{aligned} g_R &: \text{Alg}_{\mathbb{C}} \longrightarrow \text{Set}^{\circ} \\ S &\longmapsto \text{Hom}(S, R). \end{aligned}$$

Aspect (1) is by now standard in algebraic geometry. It allows one to define the various *local geometric properties* of a “space” via algebra-homomorphisms; for example, Definition 1.1. It suggests one to think of X as a sheaf over $\text{AffineSpace}_{\mathbb{C}}$. Thus, after the notion of *coverings* and *gluings* is selected, it allows one to extend the notion of a noncommutative space to that of a “*noncommutative stack*”. Aspect (2) is especially akin to our thought on D-branes. It says, in particular, that the geometry of $X = \text{Space } R$ can be revealed through an \mathbb{C} -subalgebra of R .

Example 1.4. [Azumaya point]. Consider the *Azumaya point of rank r* : $\text{Space } M_r(\mathbb{C})$. Its only two-sided prime ideal is (0) , the zero ideal. Thus, naively, one would expect $\text{Space } M_r(\mathbb{C})$ to behave like a point with an Artin \mathbb{C} -algebra as its function ring. *However*, for example, from the \mathbb{C} -algebra monomorphism $\times^r \mathbb{C} \hookrightarrow M_r(\mathbb{C})$ with image the diagonal matrices in $M_r(\mathbb{C})$, one sees that $\text{Space } M_r(\mathbb{C})$ - which is topologically a one-point set if one adopts its interpretation as $\text{Spec } M_r(\mathbb{C})$ - can dominate $\amalg_r \text{Spec } \mathbb{C}$ - which is topologically a disjoint union of r -many points -. Furthermore, consider, for example, the morphism $\varphi : \text{Space } M_r(\mathbb{C}) \rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{C}[z]$ defined by $\varphi^{\sharp} : \mathbb{C}[z] \rightarrow M_r(\mathbb{C})$ with $\varphi^{\sharp}(z) = m$ that is diagonalizable with r distinct eigenvalues $\lambda_1, \dots, \lambda_r$. Then $\text{Im } \varphi$ is a collection of r -many \mathbb{C} -points on \mathbb{A}^1 , located at $z = \lambda_1, \dots, \lambda_r$ respectively. In other words, the Azumaya noncommutativity cloud $M_r(\mathbb{C})$ over the seemingly one-point space $\text{Space } M_r(\mathbb{C})$ can really “split and condense” to a collection of concrete geometric points! Cf. FIGURE 1-1. See [L-Y1: Sec. 4.1] for more examples. Such phenomenon generalizes to Azumaya schemes; in particular, see [L-L-S-Y] for the case of Azumaya curves.

Definition 1.5. [surrogate associated to morphism]. Given $X = \text{Space } R$, let $R' \hookrightarrow R$ be a \mathbb{C} -subalgebra of R . Then, the space $X' := \text{Space } R'$ is called a *surrogate* of X . By definition, there is a built-in dominant morphism $X \rightarrow X'$, defined by the inclusion $R' \hookrightarrow R$. Given a morphism $\varphi : \text{Space } R \rightarrow \text{Space } S$ defined by $\varphi^{\sharp} : S \rightarrow R$, then $\text{Space } R_{\varphi}$, where R_{φ} is the image $\varphi^{\sharp}(S)$ of S in R , is called the *surrogate of X associated to φ* .

As Example 1.4 illustrates, commutative surrogates may be used to manifest/reveal the hidden geometry of a noncommutative space.

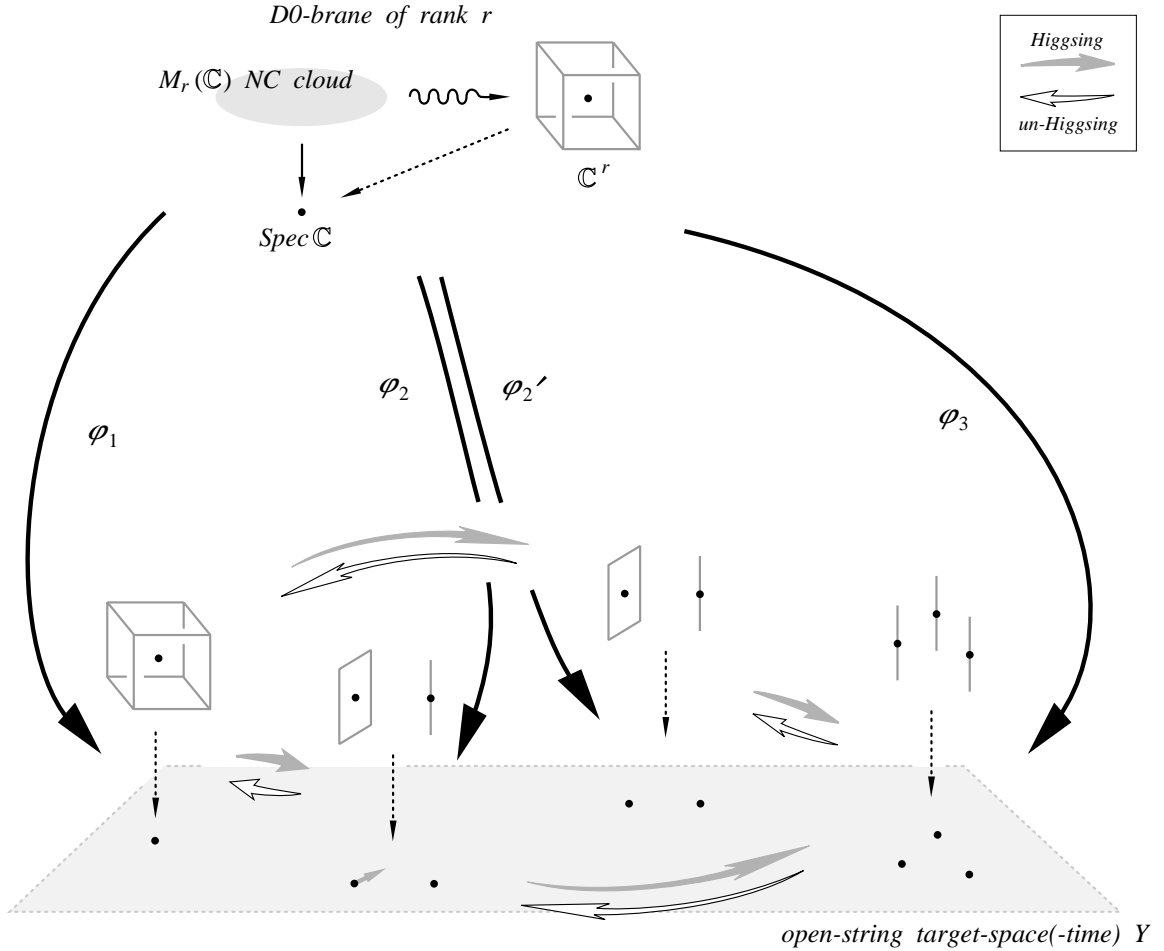


FIGURE 1-1. ([L-L-S-Y: FIGURE 2-1-1].) Despite that $Space M_r(\mathbb{C})$ may look only one-point-like, under morphisms the Azumaya “noncommutative cloud” $M_r(\mathbb{C})$ over $Space M_r(\mathbb{C})$ can “split and condense” to various schemes with a rich geometry. The latter schemes can even have more than one component. The Higgsing/un-Higgsing behavior of the Chan-Paton module of D0-branes on Y occurs due to the fact that when a morphism $\varphi : Space M_r(\mathbb{C}) \rightarrow Y$ deforms, the corresponding push-forward $\varphi_* \mathbb{C}^r$ of the fundamental module \mathbb{C}^r on $Space M_r(\mathbb{C})$ can also change/deform. These features generalize to morphisms from Azumaya schemes to Y . Here, a module over a scheme is indicated by a dotted arrow $\cdots \rightarrow$.

Definition 1.6. [push-forward of module]. Given a morphism $\varphi : X = \text{Space } R \rightarrow Y = \text{Space } S$, defined by $\varphi^\sharp : S \rightarrow R$, and a (left) R -module M , the *push-forward* of M from X to Y under φ , in notation $\varphi_* M$ or $_S M$ when φ is understood, is defined to be M as a (left) S -module via φ^\sharp . Since $\text{Ker } \varphi^\sharp \cdot M = 0$, we say that the S -module $\varphi_* M$ on Y is *supported* on $\varphi(X) \subset Y$.

In particular, any R -module M on $X = \text{Space } R$ has a push-forward on any surrogate of X .

D-branes in an affine noncommutative space à la Polchinski-Grothendieck Ansatz.

A *D-brane* is geometrically a locus in space-time that serves as the boundary condition for open strings.⁹ Through this, open strings dictate also the fields and their dynamics on D-branes. In particular, when a collection of D-branes are stacked together, the fields on the D-brane that govern the deformation of the brane are enhanced to matrix-valued, cf. Polchinski in [Po: vol. I, Sec. 8.7]. This open-string-induced phenomenon on D-branes, when re-read from Grothendieck's contravariant equivalence between the category of geometries and the category of algebras, says that D-brane world-volume carries an Azumaya-type noncommutative structure. I.e.

- *Polchinski-Grothendieck Ansatz*: D-brane has a geometry that is generically locally associated to algebras of the form $M_r(R_0)$, where R_0 is an \mathbb{R} -algebra.

See [L-Y1: Sec. 2.2] for detailed explanations.

For this work, we will be restricting ourselves to affine situations in noncommutative algebraic geometry with R_0 a commutative Noetherian \mathbb{C} -algebra. Thus:

Definition 1.7. [affine D-brane in affine target]. A *D-brane* (or *D-brane world-volume*) in an affine noncommutative space $Y = \text{Space } S$ is a triple that consists of

- a \mathbb{C} -algebra R that is isomorphic to $M_r(R_0)$ for an R_0 ,
- a (left) generically simple R -module M , which has rank r as an R_0 -module,
- a morphism $\varphi : \text{Space } R \rightarrow Y$, defined by a \mathbb{C} -algebra-homomorphism $\varphi^\sharp : S \rightarrow R$.

We will write $\varphi : (\text{Space } R, M) \rightarrow Y$ for simplicity of notations. $\varphi(X) = \text{Im } \varphi$ is called the *image-brane* on Y . M is called the *fundamental module* on $\text{Space } R$ and the push-forward $\varphi_* M$ is called the *Chan-Paton module* on the image-brane $\varphi(X)$.

Definition/Example 1.8. [D0-brane as morphism from Azumaya point with fundamental module]. A *D0-brane of length r* on an affine noncommutative space $Y = \text{Space } S$ is given by a morphism $\varphi : (\text{Space } \text{End}(V), V) \rightarrow Y$, where $V \simeq \mathbb{C}^r$. In other words, a D0-brane on Y is given by

- a finite-dimensional \mathbb{C} -vector space V and a \mathbb{C} -algebra-homomorphism: $\varphi^\sharp : S \rightarrow \text{End}(V)$.

⁹This is how one would think of a D-brane to begin with. Later development of string theory enlarges this picture considerably. See [L-Y1: References] to get a glimpse.

This is precisely a realization of a finite-dimensional \mathbb{C} -vector space V as an S -module.¹⁰ A morphism from $\varphi_1 : (\text{Space } \text{End}(V_1), V_1) \rightarrow Y$ to $\varphi_2 : (\text{Space } \text{End}(V_2), V_2) \rightarrow Y$ is a \mathbb{C} -vector-space isomorphism $h : V_2 \xrightarrow{\sim} V_1$ such that the following diagram commutes

$$\begin{array}{ccc} \text{End}(V_1) & \xleftarrow{\varphi_1^\#} & S \\ \uparrow h & \swarrow \varphi_2^\# & \\ \text{End}(V_2) & & \end{array} .$$

Here, the h -induced isomorphism $\text{End}(V_2) \xrightarrow{\sim} \text{End}(V_1)$ is also denoted by h . In other words, a morphism between φ_1 and φ_2 is an isomorphism of the corresponding V_1 and V_2 as S -modules.

It follows from the above definition/example that the moduli stack $\mathfrak{M}_r^{\text{D0}}(Y)$ of D0-branes of length r on $Y = \text{Space } S$ has an atlas given by the *representation scheme* $\text{Rep}(S, M_r(\mathbb{C}))$ that parameterizes all \mathbb{C} -algebra-homomorphisms $S \rightarrow M_r(\mathbb{C})$. The latter commutative scheme serves also as the moduli space of morphisms $\text{Space } M_r(\mathbb{C}) \rightarrow Y$ with $M_r(\mathbb{C})$ treated as fixed. From [K-R] and [leB1], one expects that noncommutative geometric structures/properties of $Y = \text{Space } S$ are reflected in properties/structures of the discrete family of commutative schemes $\text{Rep}(S, M_r(\mathbb{C}))$, $r \in \mathbb{Z}_{>0}$. This anticipation from noncommutative algebraic geometry rings hand in hand with the stringy philosophy to use D-branes as a probe to the nature of space-time!

2 Deformations of a conifold via an Azumaya probe.

Using a toric setup for a conifold that is meant to match Klebanov-Witten [K-W], we discuss how an Azumaya probe “sees” deformations of the conifold in a way that resembles Klebanov-Strassler [K-S].

A toric setup for the standard local conifold.

The standard local conifold $Y = \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4))$ can be given an affine toric variety description as follows. Let $N = \bigoplus_{i=1}^4 \mathbb{Z} e_i$ be the rank 4 lattice and Δ be the fan in N that consists of the single non-strongly convex polyhedral cone $\sigma = \bigoplus_{i=1}^6 \mathbb{R}_{\geq 0} v_i$ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, where

$$\begin{aligned} v_1 &= e_1, & v_2 &= e_2, & v_3 &= e_3, & v_4 &= -e_1 + e_2 + e_3, \\ v_5 &= e_1 - e_2 - e_3 + e_4, & v_6 &= -v_5 = -e_1 + e_2 + e_3 - e_4. \end{aligned}$$

Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice of N , with the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$. Then, the dual cone σ^\vee of σ is given by $\text{Span}_{\mathbb{R}_{\geq 0}}\{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\} \subset M_{\mathbb{R}}$. This determines a commutative semigroup

$$S_\sigma = \sigma^\vee \cap M = \text{Span}_{\mathbb{Z}_{\geq 0}}\{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\}$$

¹⁰Thus, a D0-brane on $\text{Space } S$ is precisely an S -module that is of finite dimension as a \mathbb{C} -vector space. Such a direct realization of a D-brane as a module on a target-space is a special feature for D0-branes. For high dimensional D-branes, such modules on the target-space give only a subclass of D-branes that describe solitonic branes in space-time.

with generators $e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*$. The corresponding group-algebra

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\xi_1\xi_2, \xi_3\xi_4, \xi_1\xi_3, \xi_2\xi_4] \subset \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4],$$

where $\xi_i = \exp(e_i^*)$, $i = 1, 2, 3, 4$, defines then the conifold

$$Y = U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)),$$

where

$$z_1 = \xi_1\xi_2, \quad z_2 = \xi_3\xi_4, \quad z_3 = \xi_1\xi_3, \quad z_4 = \xi_2\xi_4.$$

Note that built into this construction is the morphism

$$\mathbb{A}_{[\xi_1, \xi_2, \xi_3, \xi_4]}^4 := \text{Spec}(\mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4]) \longrightarrow Y \hookrightarrow \mathbb{A}_{[z_1, z_2, z_3, z_4]}^4 := \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]),$$

where the first morphism is surjective.

An Azumaya probe to a noncommutative space and its commutative descent.

Guided by [K-W] and [K-S], where ξ_i 's here play the role of scalar component of chiral superfields involved in ibidem, consider the noncommutative space

$$\begin{aligned} \Xi &:= \text{Space}(R_\Xi) \\ &:= \text{Space}\left(\frac{\mathbb{C}\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle}{([\xi_1\xi_3, \xi_2\xi_4], [\xi_1\xi_3, \xi_1\xi_4], [\xi_1\xi_3, \xi_2\xi_3], [\xi_2\xi_4, \xi_1\xi_4], [\xi_2\xi_4, \xi_2\xi_3], [\xi_1\xi_4, \xi_2\xi_3])}\right), \end{aligned}$$

where $\mathbb{C}\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle$ is the associative unital \mathbb{C} -algebra generated by $\{\xi_1, \xi_2, \xi_3, \xi_4\}$, (\dots) in the denominator is the two-sided ideal generated by \dots , and $[\bullet, \bullet']$ is the commutator. Here, $\text{Space}(\bullet)$ is the would-be space associated to the ring \bullet . We do not need its detail as all we need are morphisms between spaces which can be contravariantly expressed as ring-homomorphisms. By construction, the scheme-morphism $\mathbb{A}_{[\xi_1, \xi_2, \xi_3, \xi_4]}^4 \rightarrow \mathbb{A}_{[z_1, z_2, z_3, z_4]}^4$, whose image is Y , extends to a morphism

$$\pi^\Xi : \Xi \longrightarrow \mathbb{A}_{[z_1, z_2, z_3, z_4]}^4,$$

whose image is now the whole $\mathbb{A}_{[z_1, z_2, z_3, z_4]}^4$. The underlying ring-homomorphism is given by

$$\begin{aligned} \pi^{\Xi, \sharp} : \mathbb{C}[z_1, z_2, z_3, z_4] &\longrightarrow R_\Xi \\ z_1 &\longmapsto \xi_1\xi_3 \\ z_2 &\longmapsto \xi_2\xi_4 \\ z_3 &\longmapsto \xi_1\xi_4 \\ z_4 &\longmapsto \xi_2\xi_3. \end{aligned}$$

Consider a D0-brane moving on the conifold Y via the chiral superfields. In terms of Polchinski-Grothendieck Ansatz, this is realized by the descent of morphisms $\tilde{\varphi} : \text{Space } M_1(\mathbb{C}) = \text{Spec } \mathbb{C} \rightarrow \Xi$ to $\varphi : \text{Space } M_1(\mathbb{C}) = \text{Spec } \mathbb{C} \rightarrow Y$ by the specification of ring-homomorphisms

$$\tilde{\varphi}^\sharp : \xi_1 \longmapsto a_1; \quad \xi_2 \longmapsto a_2; \quad \xi_3 \longmapsto b_1; \quad \xi_4 \longmapsto b_2.$$

The corresponding

$$\varphi^\sharp : z_1 \longmapsto a_1b_1; \quad z_2 \longmapsto a_2b_2; \quad z_3 \longmapsto a_1b_2; \quad z_4 \longmapsto a_2b_1$$

gives a morphism $\varphi : \text{Spec } \mathbb{C} \rightarrow Y$, i.e. a \mathbb{C} -point on the conifold Y .

Deformations of the conifold via an Azumaya probe: descent of noncommutative superficially-infinitesimal deformations.

We now consider what happens if we add a D0-brane to the conifold point of Y . This D0-brane together with the D0-brane probe is the image of a morphism from the Azumaya point $Space M_2(\mathbb{C})$ to Y . Thus we should consider morphisms $\tilde{\varphi} : Space M_2(\mathbb{C}) \rightarrow \Xi$ of noncommutative spaces and their descent φ on related commutative spaces.

Definition 2.1. [superficially infinitesimal deformation]. Given finitely-presented associative unital rings, $R = \langle r_1, \dots, r_m \rangle / \sim$ and S , and a ring-homomorphism $h : R \rightarrow S$. A *superficially infinitesimal deformation of h with respect to the generators $\{r_1, \dots, r_m\}$ of R* is a ring-homomorphism $h_\varepsilon : R \rightarrow S$ such that $h_\varepsilon(r_i) = h(r_i) + \varepsilon_i$ with $\varepsilon_i^2 = 0$, for $i = 1, \dots, m$.

Remark 2.2. [commutative S]. Note that when S is commutative, a superficially infinitesimal deformation of $h_\varepsilon : R \rightarrow S$ is an infinitesimal deformation of h in the sense that $h_\varepsilon(r) = h(r) + \varepsilon_r$ with $(\varepsilon_r)^2 = 0$, for all $r \in R$. This is no longer true for general noncommutative S .

To begin, consider the diagram of morphisms of spaces

$$\begin{array}{ccc} Space M_2(\mathbb{C}) & \xrightarrow{\tilde{\varphi}} & \Xi = Space R_\Xi \\ \parallel & & \downarrow \pi^\Xi \\ Space M_2(\mathbb{C}) & \xrightarrow{\varphi} & Y \hookrightarrow \mathbb{A}^4 \end{array}$$

given by ring-homomorphisms

$$\begin{array}{ccc} M_2(\mathbb{C}) & \xleftarrow{\tilde{\varphi}^\#} & R_\Xi \\ \parallel & & \uparrow \pi^{\Xi, \#} \\ M_2(\mathbb{C}) & \xleftarrow{\varphi^\#} & \mathbb{C}[z_1, z_2, z_3, z_4] / (z_1 z_2 - z_3 z_4) \leftarrow \mathbb{C}[z_1, z_2, z_3, z_4] \end{array}$$

with

$$A_1; A_2; B_1; B_2 \xleftarrow{\tilde{\varphi}^\#} \xi_1; \xi_2; \xi_3; \xi_4$$

$$\begin{array}{ccc} & & \xi_1 \xi_3; \xi_2 \xi_4; \xi_1 \xi_4; \xi_2 \xi_3 \\ & & \uparrow \pi^{\Xi, \#} \\ A_1 B_1; A_2 B_2; A_1 B_2; A_2 B_1 & \xleftarrow{\varphi^\#} & \overline{z_1}; \overline{z_2}; \overline{z_3}; \overline{z_4} \leftarrow z_1; z_2; z_3; z_4 \end{array}$$

where

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

The image D-brane $\varphi(Space M_2(\mathbb{C}))$ is supported on a subscheme Z of Y associated to the ideal

$$Ker \varphi = \begin{cases} (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) \cap (\overline{z_1} - a_1 b_1, \overline{z_2} - a_2 b_2, \overline{z_3} - a_1 b_2, \overline{z_4} - a_2 b_1) \\ \quad \text{if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) \neq (0, 0, 0, 0), \\ (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) \quad \text{if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) = (0, 0, 0, 0). \end{cases}$$

The former corresponds to two simple non-coincident D0-branes, each with Chan-Paton module \mathbb{C} , on the conifold Y with one of them sitting at the conifold point $\mathbf{0}$ and the other sitting at the \mathbb{C} -point with the coordinate tuple $(a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1)$ while the latter corresponds to coincident D0-branes at $\mathbf{0}$ with the Chan-Paton module enhanced to \mathbb{C}^2 at $\mathbf{0}$. In both situations, the support Z of the D-brane is reduced. This is the transverse-to-the-effective-space-time part of the D3-brane setting in [K-W] and [K-S].

Consider now a superficially infinitesimal deformation of $\tilde{\varphi}$ given by:

$$\begin{array}{ccc} \text{Space } M_2(\mathbb{C}) & \xrightarrow{\tilde{\varphi}_{(\delta_1, \delta_2, \eta_1, \eta_2)}} & \Xi = \text{Space } R_\Xi \\ M_2(\mathbb{C}) & \xleftarrow{\tilde{\varphi}^\#_{(\delta_1, \delta_2, \eta_1, \eta_2)}} & R_\Xi \\ A_1; A_2; B_1; B_2 & \longleftarrow & \xi_1; \xi_2; \xi_3; \xi_4 \end{array}$$

where

$$A_1 = \begin{bmatrix} a_1 & \delta_1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & \delta_2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 & 0 \\ \eta_1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 & 0 \\ \eta_2 & 0 \end{bmatrix}.$$

Should $\text{Space } M_2(\mathbb{C})$ be a commutative space, this would give only an infinitesimal deformation of φ . However, $\text{Space } M_2(\mathbb{C})$ is not a commutative space and, hence, the naive anticipation above could fail. Indeed, the descent $\varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}$ of $\tilde{\varphi}_{(\delta_1, \delta_2, \eta_1, \eta_2)}$ is given by

$$\begin{array}{ccc} \text{Space } M_2(\mathbb{C}) & \xrightarrow{\varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}} & \mathbb{A}^4 \\ M_2(\mathbb{C}) & \xleftarrow{\varphi^\#_{(\delta_1, \delta_2, \eta_1, \eta_2)}} & \mathbb{C}[z_1, z_2, z_3, z_4] \\ A_1 B_1; A_2 B_2; A_1 B_2; A_2 B_1 & \longleftarrow & z_1; z_2; z_3; z_4, \end{array}$$

i.e.

$$\begin{array}{c} \begin{bmatrix} a_1 b_1 + \delta_1 \eta_1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_2 b_2 + \delta_2 \eta_2 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_1 b_2 + \delta_1 \eta_2 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_2 b_1 + \delta_2 \eta_1 & 0 \\ 0 & 0 \end{bmatrix} \\ \longleftarrow z_1; z_2; z_3; z_4. \end{array}$$

The image $Z := \varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}(\text{Space } M_2(\mathbb{C}))$ of the Azumaya point $\text{Space } M_2(\mathbb{C})$ under $\varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}$ remains a 0-dimensional reduced scheme, consisting of either two \mathbb{C} -points - with one of them at $\mathbf{0}$ - or $\mathbf{0}$ alone. However,

$$z_1 z_2 - z_3 z_4 = \begin{vmatrix} z_1 & z_3 \\ z_4 & z_2 \end{vmatrix} = \begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$$

vanishes if and only if either $\begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix}$ or $\begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$ is 0. In other words, while the image $\varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}(\text{Space } M_2(\mathbb{C}))$ still contains the conifold-point $\mathbf{0}$ in Y , as a whole it may longer lie completely even in any infinitesimal neighborhood of the conifold Y in \mathbb{A}^4 . I.e.:

Lemma 2.3. [deformation from descent of superficially infinitesimal deformation].
The descent $\varphi_{(\delta_1, \delta_2, \eta_1, \eta_2)}$ of a superficially infinitesimal deformation of $\tilde{\varphi}$ can truly deform φ . Thus, an appropriate choice of a subspace of the space of morphisms $\tilde{\varphi}_{(\bullet)} : \text{Space } M_2(\mathbb{C}) \rightarrow \Xi$ can descend to give a space of morphisms $\varphi_{(\bullet)} : \text{Space } M_2(\mathbb{C}) \rightarrow \mathbb{A}^4$ that is parameterized by a deformed conifold Y' .

This realizes a deformed conifold as a moduli space of morphisms from an Azumaya point and is the reason why the Azumaya probe can see a deformation of the conifold Y from the viewpoint of Polchinski-Grothendieck Ansatz. FIGURE 2-1.

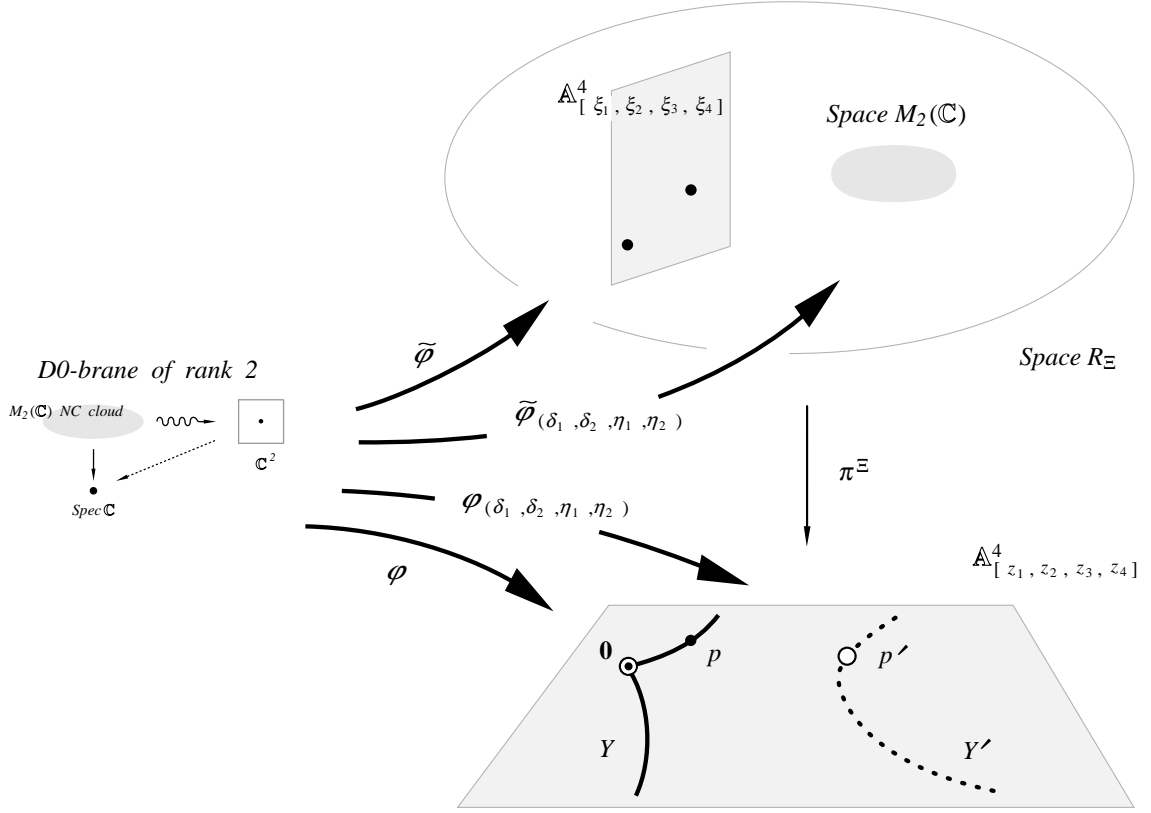


FIGURE 2-1. A generic superficially infinitesimal deformation $\tilde{\varphi}_{(\delta_1, \delta_2, \eta_1, \eta_2)}$ of $\tilde{\varphi}$ has a noncommutative image $\simeq \text{Space } M_2(\mathbb{C})$. It then descends to $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ and becomes a pair of \mathbb{C} -points on $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$. One of the points is the conifold singularity $\mathbf{0} = V(z_1, z_2, z_3, z_4) \in Y$ and the other is the point $p' = V(z_1 - a_1 b_1 - \delta_1 \eta_1, z_2 - a_2 b_2 - \delta_2 \eta_2, z_3 - a_1 b_2 - \delta_1 \eta_2, z_4 - a_2 b_1 - \delta_2 \eta_1)$ off Y (generically). Through such deformations, any \mathbb{C} -point on $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ can be reached. Thus, one can realize a deformation Y' of Y in $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ by a subvariety in $\text{Rep}(R_\Xi, M_2(\mathbb{C}))$. This is the Azumaya-geometry origin of the phenomenon in Klebanov-Strassler [K-S] that a trapped D-brane sitting on the conifold singularity may give rise to a deformation of the moduli space of SQFT on the D3-brane probe, turning a conifold to a deformed conifold. Our D0-brane here corresponds to the internal part of the effective-space-time-filling D3-brane world-volume of [K-S].

Remark 2.4. [generalization]. This phenomenon can be generalized beyond a conifold. In particular, recall that an A_n -singularity on a complex surface is also a toric singularity. Similar mechanism/discussion can be applied to deform a transverse A_n -singularity via morphisms from an Azumaya probe.

Deformations of the conifold via an Azumaya probe: details.

We now give an explicit construction that realizes Lemma 2.3. For convenience¹¹, we will take $Space M_2(\mathbb{C})$ as fixed, and is equipped with the defining fundamental (left) $M_2(\mathbb{C})$ -module \mathbb{C}^2 . Then, the space $Mor^a(Space M_2(\mathbb{C}), \Xi)$ of admissible morphisms of the form $\tilde{\varphi}_{(\bullet)}$ in the previous theme is naturally realized as a subscheme $Rep^a(R_\Xi, M_2(\mathbb{C}))$ of the representation scheme $Rep(R_\Xi, M_2(\mathbb{C}))$ that parameterizes elements in $Mor_{\mathbb{C}\text{-Alg}}(R_\Xi, M_2(\mathbb{C}))$. From the previous discussion,

$$\begin{aligned} Rep^a(R_\Xi, M_2(\mathbb{C})) &= Spec \mathbb{C}[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2] \\ &=: \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 = \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2]}^4 \times_{\mathbb{C}} \mathbb{A}_{[b_1, b_2, \eta_1, \eta_2]}^4. \end{aligned}$$

Consider also the space $Mor^a(Space M_2(\mathbb{C}), \mathbb{A}^4)$ of morphisms from Azumaya point to $\mathbb{A}_{[z_1, z_2, z_3, z_4]}^4$ with the associated \mathbb{C} -algebra-homomorphism of the form

$$z_1 \mapsto \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_2 \mapsto \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_3 \mapsto \begin{bmatrix} c_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_4 \mapsto \begin{bmatrix} c_4 & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote the associated representation scheme by

$$Rep^a(\mathbb{C}[z_1, z_2, z_3, z_4], M_2(\mathbb{C})), \quad \text{which is} \quad Spec \mathbb{C}[c_1, c_2, c_3, c_4] =: \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4.$$

The \mathbb{C} -algebra homomorphism $\pi^{\Xi, \sharp} : \mathbb{C}[z_1, z_2, z_3, z_4] \rightarrow R_\Xi$ induces a morphism of representation schemes

$$\pi_{Rep} : \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 \longrightarrow \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$$

with π_{Rep}^\sharp given in a matrix form by

$$\pi_{Rep}^\sharp : \begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{bmatrix}.$$

Lemma 2.5. [enough superficially infinitesimally deformed morphisms].

$$\pi_{Rep} : \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 \longrightarrow \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$$

is surjective.

There are three homeomorphism classes of fibers of π_{Rep} over a closed point of $\mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$, depending on the rank of $\begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix}$.

Lemma 2.6. [topological type of fibers of π_{Rep}]. *Let $C_{[c_1, c_2, c_3, c_4]}^3$ be the subvariety of $\mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$ associated to the ideal $(c_1 c_2 - c_3 c_4)$. Similarly, for $C_{[a_1, a_2, \delta_1, \delta_2]}^3$ and $C_{[b_1, b_2, \eta_1, \eta_2]}^3$. Then:*

- (0) *Over $\mathbf{0}$, the fiber is given by $\mathbb{A}_{[a_1, a_2, \delta_1, \delta_2]}^4 \cup \mathbb{A}_{[b_1, b_2, \eta_1, \eta_2]}^4 \cup \Pi^5$, where Π^5 is a 5-dimensional irreducible affine scheme meeting $\mathbb{A}_{[a_1, a_2, \delta_1, \delta_2]}^4 \cup \mathbb{A}_{[b_1, b_2, \eta_1, \eta_2]}^4$ along $C_{[a_1, a_2, \delta_1, \delta_2]}^3 \cup C_{[b_1, b_2, \eta_1, \eta_2]}^3$.*

¹¹If $Space M_2(\mathbb{C})$ is not fixed, then one studies Artin stacks that parameterizes morphisms in question from $Space M_2(\mathbb{C})$ to $Space R_\Xi$, the conifold Y , and $\mathbb{A}_{[z_1, z_2, z_3, z_4]}^4$ respectively. The discussion given here is then on an atlas of the stack in question.

- (1) Over a closed point of $C_{[c_1, c_2, c_3, c_4]}^3 - \{\mathbf{0}\}$, the fiber is the union $\Pi_1^4 \cup \Pi_2^4$ of two irreducible 4-dimensional affine scheme meeting at a deformed conifold.
- (2) Over a closed point of $\mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 - C_{[c_1, c_2, c_3, c_4]}^3$, the fiber is isomorphic to $\mathbb{A}_{[a_1, a_2, \delta_1, \delta_2]}^4 - C_{[a_1, a_2, \delta_1, \delta_2]}^3 \simeq \mathbb{A}_{[b_1, b_2, \eta_1, \eta_2]}^4 - C_{[b_1, b_2, \eta_1, \eta_2]}^3$.

The lemma follows from a straightforward computation.¹² Note that the fundamental group as an analytic space is given by

$$\begin{aligned} \pi_1(\mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 - C_{[c_1, c_2, c_3, c_4]}^3) &\simeq \pi_1(\mathbb{A}_{[a_1, a_2, \delta_1, \delta_2]}^4 - C_{[a_1, a_2, \delta_1, \delta_2]}^3) \\ &\simeq \pi_1(\mathbb{A}_{[b_1, b_2, \eta_1, \eta_2]}^4 - C_{[b_1, b_2, \eta_1, \eta_2]}^3) \simeq \mathbb{Z} \end{aligned}$$

and that the smooth bundle-morphism

$$\pi_{Rep} : \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 - \pi_{Rep}^{-1}(C_{[c_1, c_2, c_3, c_4]}^3) \longrightarrow \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 - C_{[c_1, c_2, c_3, c_4]}^3$$

exhibits a monodromy behavior which resembles that of a Dehn twist.

The map $\pi_{Rep} : \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 \rightarrow \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$ admits sections, i.e. morphism $s : \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 \rightarrow \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8$ such that $\pi_{Rep} \circ s = \text{the identity map on } \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$.

Example 2.7. [section of π_{Rep}]. Let $t \in GL_2(\mathbb{C})$, then a simple family of sections of π_{Rep}

$$s_t : \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 \longrightarrow \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8$$

is given compactly in a matrix expression by (with t also in its defining 2×2 -matrix form)

$$s_t^\# : \left(\begin{bmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{bmatrix} \right) \longmapsto \left(\begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix} \cdot t^{-1}, t \right).$$

Through any section $s : \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4 \rightarrow \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8$, one can realize $Y' \amalg \{\mathbf{0}\}$, where Y' is a deformation of the conifold Y in $\mathbb{A}^4 = \mathbb{A}_{[z_1, z_2, z_3, z_4]}^4$ and $\mathbf{0}$ is the singular point on Y , as the descent of a family of superficially infinitesimal deformations of morphisms from Azumaya point to the noncommutative space Ξ . In string theory words,

- *deformations of a conifold via a D-brane probe are realized by turning on D-branes at the singularity appropriately; the conifold is deformed and becomes smooth while leaving the trapped D-branes at the singularity behind.*

Cf. FIGURE 2-1.

¹²It is very instructive to think of the fibration $\pi_{Rep} : \mathbb{A}_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]}^8 \rightarrow \mathbb{A}_{[c_1, c_2, c_3, c_4]}^4$ as defining a one-matrix-parameter family of “matrix nodal curves” in the sense of noncommutative geometry.

3 Resolutions of a conifold via an Azumaya probe.

In this section, we consider resolutions of the conifold $Y = \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4))$ from the viewpoint of an Azumaya probe. Recall the following diagram of resolutions of Y from blow-ups of Y :

$$\begin{array}{ccc} & \tilde{Y} & \\ f_+ \swarrow & \downarrow \pi & \searrow f_- \\ Y_+ & & Y_- \\ \pi_+ \swarrow & & \swarrow \pi_- \\ & Y & \end{array},$$

where

- $\pi : \tilde{Y} = \text{Bl}_{V(I)} Y = \text{Proj}(\oplus_{i=0}^{\infty} I^i) \rightarrow Y$ with $I = (z_1, z_2, z_3, z_4)$,
- $\pi_+ : Y_+ = \text{Bl}_{V(I_+)} Y = \text{Proj}(\oplus_{i=0}^{\infty} I_+^i) \rightarrow Y$ with $I_+ = (z_1, z_3)$, and
- $\pi_- : Y_- = \text{Bl}_{V(I_-)} Y = \text{Proj}(\oplus_{i=0}^{\infty} I_-^i) \rightarrow Y$ with $I_- = (z_1, z_4)$

are blow-ups of Y along the specified subschemes $V(\bullet)$ associated respectively to the ideals I , I_+ , and I_- of $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$ as given. Here, we set $I_{(\pm)}^0 = \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$. Let $\mathbf{0} = V(z_1, z_2, z_3, z_4)$ be the singular point of Y . Then the exceptional locus in each case is given respectively by $\pi^{-1}(\mathbf{0}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\pi_+^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$, and $\pi_-^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$; Y_+ and Y_- as schemes/ Y are related by a flop; and the restriction of birational morphisms $f_{\pm} : \tilde{Y} \rightarrow Y_{\pm}$ to $\pi^{-1}(\mathbf{0})$ corresponds to the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to each of its two factors.

D-brane probe resolutions of a conifold via the Azumaya structure.

An atlas for the stack of morphisms from $\text{Space } M_2(\mathbb{C})$ to Y is given by the representation scheme $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ with the $PGL_2(\mathbb{C})$ -action induced from the $GL_2(\mathbb{C})$ -action on the fundamental module \mathbb{C}^2 . For convenience, we will also call this a $GL_2(\mathbb{C})$ -action on $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Let

$$W = \text{Rep}^{\text{singleton}}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$$

be the subscheme of $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ that parameterizes D0-branes $\varphi : (\text{Spec } \mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^2) \rightarrow Y$ with $(\text{Im } \varphi)_{\text{red}}$ a single \mathbb{C} -point on Y . Explicitly, W is the image scheme of

$$GL_2(\mathbb{C}) \times W_{ut} \longrightarrow \text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$$

where

$$\begin{aligned} W_{ut} &= \left\{ \rho : \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \rightarrow M_2(\mathbb{C}) \mid \rho(z_i) \text{ is of the form } \begin{bmatrix} a_i & \varepsilon_i \\ 0 & a_i \end{bmatrix} \right\} \\ &\subset \text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C})) \end{aligned}$$

and the morphism \longrightarrow is from the restriction of the $GL_2(\mathbb{C})$ -group on $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Using this notation, as a scheme,

$$\begin{aligned} W_{ut} &= \text{Spec}(\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]/(a_1 a_2 - a_3 a_4, a_2 \varepsilon_1 + a_1 \varepsilon_2 - a_4 \varepsilon_3 - a_3 \varepsilon_4)) \\ &\subset \text{Spec}(\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]) =: \mathbb{A}_{[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]}^8. \end{aligned}$$

Imposing the trivial $GL_2(\mathbb{C})$ -action on Y , then by construction, there is a natural $GL_2(\mathbb{C})$ -equivariant morphism

$$\pi^W : W \longrightarrow Y$$

defined by $\pi^{W,\#}(z_i) = \frac{1}{2} \text{Tr} \rho(z_i) = a_i$ in the above notation. This is the morphism that sends a $\varphi : (\text{Spec } \mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^2) \rightarrow Y$ under study to $(\text{Im } \varphi)_{\text{red}} \in Y$.

Lemma 3.1. [Azumaya probe to conifold singularity]. *There exists $GL_2(\mathbb{C})$ -invariant subschemes \tilde{Y}' , Y'_+ , and Y'_- of W such that their geometric quotient $\tilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ under the $GL_2(\mathbb{C})$ -action exist and are isomorphic to \tilde{Y} , Y_+ , and Y_- respectively. Furthermore, under these isomorphisms, the restriction of $\pi^W : W \rightarrow Y$ to \tilde{Y}' , Y'_+ , and Y'_- descends to morphisms from the quotient spaces $\tilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ to Y that realize the resolution diagram*

$$\begin{array}{ccc} & \tilde{Y} & \\ f_+ \swarrow & \downarrow \pi & \searrow f_- \\ Y_+ & & Y_- \\ \pi_+ \searrow & & \swarrow \pi_- \\ & Y & \end{array}$$

of Y at the beginning of this section.

It is in the sense of the above lemma we say that

- an Azumaya point of rank ≥ 2 and hence a D -brane probe of multiplicity ≥ 2 can “see” all the three different resolutions of the conifold singularity.

It should also be noted that Lemma 3.1 is a special case of a more general statement that reflects the fact that the stack of morphisms from Azumaya points to a (general, possibly singular, Noetherian) scheme Y is a generalization of the notion of jet-schemes of Y . Cf. [L-Y2: Figure 0-1, caption].

An explicit construction of \tilde{Y}' , Y'_+ , and Y'_- .

An explicit construction of \tilde{Y}' , Y'_+ , and Y'_- , and hence the proof of Lemma 3.1, follows from a lifting-to- W of an affine atlas of $\text{Proj}(\oplus_{i=0}^{\infty} I^i_{(\pm)})$.

To construct \tilde{Y}' , recall that $I = (z_1, z_2, z_3, z_4)$. An affine atlas of \tilde{Y} is given by the collection

$$U^{(z_i)} = \text{Spec}((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \simeq \begin{cases} \text{Spec}(\mathbb{C}[z_1, u_2, u_3, u_4]/(u_2 - u_3 u_4)) \simeq \mathbb{A}^3_{[z_1, u_3, u_4]} & \text{for } i = 1; \\ \text{Spec}(\mathbb{C}[u_1, z_2, u_3, u_4]/(u_1 - u_3 u_4)) \simeq \mathbb{A}^3_{[z_2, u_3, u_4]} & \text{for } i = 2; \\ \text{Spec}(\mathbb{C}[u_1, u_2, z_3, u_4]/(u_1 u_2 - u_4)) \simeq \mathbb{A}^3_{[u_1, u_2, z_3]} & \text{for } i = 3; \\ \text{Spec}(\mathbb{C}[u_1, u_2, u_3, z_4]/(u_1 u_2 - u_3)) \simeq \mathbb{A}^3_{[u_1, u_2, z_4]} & \text{for } i = 4. \end{cases}$$

Here, $z_i \in I$ has grade 1 and $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0$ is the grade-0 component of the graded algebra $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]$. Each $U^{(z_i)}$ is equipped with a built-in morphism $\pi^{(i)} : U^{(z_i)} \rightarrow Y$ in such a way that, when all four are put together, they glue to give the resolution $\pi : \tilde{Y} \rightarrow Y$.

Consider the lifting $\{\pi^{(i)'} : U^{(z_i)} \rightarrow W \mid i = 1, 2, 3, 4\}$ of the atlas $\{\pi^{(i)} : U^{(z_i)} \rightarrow Y \mid i = 1, 2, 3, 4\}$ of \tilde{Y} that is given by the lifting $\{\pi^{(i)'} : U^{(z_i)} \rightarrow W_{ut} \subset W \mid i = 1, 2, 3, 4\}$ defined by

$$\begin{aligned} \pi^{(1)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_1, z_1 u_2, z_1 u_3, z_1 u_4, 1, u_2, u_3, u_4 \text{ respectively,} \\ \pi^{(2)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_2 u_1, z_2, z_2 u_3, z_2 u_4, u_1, 1, u_3, u_4 \text{ respectively,} \\ \pi^{(3)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_3 u_1, z_3 u_2, z_3, z_3 u_4, u_1, u_2, 1, u_4 \text{ respectively,} \\ \pi^{(4)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_4 u_1, z_4 u_2, z_4 u_3, z_4, u_1, u_2, u_3, 1 \text{ respectively.} \end{aligned}$$

$\pi^{(i)'}$, $i = 1, 2, 3, 4$, are now embeddings into W with the property that for any geometric point $p \in U^{(z_i)} \times_{\tilde{Y}} U^{(z_j)}$, $\pi^{(i)'}(p)$ and $\pi^{(j)'}(p)$ lies in the same $GL_2(\mathbb{C})$ -orbit in W . In other words, up to the pointwise $GL_2(\mathbb{C})$ -action, they are glueable. Let \tilde{Y}' be the image scheme of the morphism

$$GL_2(\mathbb{C}) \times (U^{(z_1)} \amalg U^{(z_2)} \amalg U^{(z_3)} \amalg U^{(z_4)}) \longrightarrow W$$

via $\pi^{(1)'} \amalg \pi^{(2)'} \amalg \pi^{(3)'} \amalg \pi^{(4)'}$ and the $GL_2(\mathbb{C})$ -action on W . Then it follows that the geometric quotient $\tilde{Y}'/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $\tilde{Y}'/GL_2(\mathbb{C}) \xrightarrow{\sim} \tilde{Y}$, as schemes over Y , through the defining embeddings $U^{(z_i)} \hookrightarrow \tilde{Y}$, $i = 1, 2, 3, 4$.

For Y'_+ , recall that $I_+ = (z_1, z_3)$. An affine atlas of Y_+ is given by the collection

$$U_+^{(z_i)} = \text{Spec}((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \simeq \begin{cases} \text{Spec}(\mathbb{C}[z_1, z_2, u_3, z_4]/(z_2 - z_4 u_3)) \simeq \mathbb{A}_{[z_1, u_3, z_4]}^3 & \text{for } i = 1; \\ \text{Spec}(\mathbb{C}[u_1, z_2, z_3, z_4]/(z_2 u_1 - z_4)) \simeq \mathbb{A}_{[u_1, z_2, z_3]}^3 & \text{for } i = 3. \end{cases}$$

Each $U_+^{(z_i)}$ is equipped with a built-in morphism $\pi_+^{(i)} : U_+^{(z_i)} \rightarrow Y$ in such a way that, when both are put together, they glue to give the resolution $\pi_+ : Y_+ \rightarrow Y$.

Consider the lifting $\{\pi_+^{(i)'} : U_+^{(z_i)} \rightarrow W \mid i = 1, 3\}$ of the atlas $\{\pi_+^{(i)} : U_+^{(z_i)} \rightarrow Y \mid i = 1, 3\}$ of Y_+ that is given by the lifting $\{\pi_+^{(i)'} : U_+^{(z_i)} \rightarrow W_{ut} \subset W \mid i = 1, 3\}$ defined by

$$\begin{aligned} \pi_+^{(1)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_1, z_4 u_3, z_1 u_3, z_4, 1, 0, u_3, 0 \text{ respectively,} \\ \pi_+^{(3)'} \iota_{\#} & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_3 u_1, z_2, z_3, z_2 u_1, u_1, 0, 1, 0 \text{ respectively.} \end{aligned}$$

The pair, $\pi_+^{(1)'}$ and $\pi_+^{(3)'}$, are now embeddings into W that, as in the case of \tilde{Y} , are glueable up to the pointwise $GL_2(\mathbb{C})$ -action. Same construction as in the case of \tilde{Y} gives then a $GL_2(\mathbb{C})$ -invariant subscheme Y'_+ of W whose geometric quotient $Y'_+/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $Y'_+/GL_2(\mathbb{C}) \xrightarrow{\sim} Y_+$ as schemes over Y .

For Y'_- , recall that $I_- = (z_1, z_4)$. The construction is identical to that in the case of Y_+ after relabelling. An affine atlas of Y_- is given by the collection

$$U_-^{(z_i)} = \text{Spec}((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \simeq \begin{cases} \text{Spec}(\mathbb{C}[z_1, z_2, z_3, u_4]/(z_2 - z_3 u_4)) \simeq \mathbb{A}_{[z_1, z_3, u_4]}^3 & \text{for } i = 1; \\ \text{Spec}(\mathbb{C}[u_1, z_2, z_3, z_4]/(z_2 u_1 - z_3)) \simeq \mathbb{A}_{[u_1, z_2, z_4]}^3 & \text{for } i = 4. \end{cases}$$

Each $U_-^{(z_i)}$ is equipped with a built-in morphism $\pi_-^{(i)} : U_-^{(z_i)} \rightarrow Y$ in such a way that, when both are put together, they glue to give the resolution $\pi_- : Y_- \rightarrow Y$.

Consider the lifting $\{\pi_-^{(i)'} : U_-^{(z_i)} \rightarrow W \mid i = 1, 4\}$ of the atlas $\{\pi_-^{(i)} : U_-^{(z_i)} \rightarrow Y \mid i = 1, 4\}$ of Y_- that is given by the lifting $\{\pi_-^{(i)'} : U_-^{(z_i)} \rightarrow W_{ut} \subset W \mid i = 1, 4\}$ defined by

$$\begin{aligned}
\pi_-^{(1)'} \# & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_1, z_3 u_4, z_3, z_1 u_4, 1, 0, 0, u_4 \text{ respectively,} \\
\pi_-^{(4)'} \# & : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \longmapsto z_4 u_1, z_2, z_2 u_1, z_4, u_1, 0, 0, 1 \text{ respectively.}
\end{aligned}$$

The pair, $\pi_-^{(1)'}$ and $\pi_-^{(4)'}$, are now embeddings into W that are glueable up to the pointwise $GL_2(\mathbb{C})$ -action. Same construction as in the case of \tilde{Y} gives then a $GL_2(\mathbb{C})$ -invariant subscheme Y'_- of W whose geometric quotient $Y'_-/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $Y'_-/GL_2(\mathbb{C}) \xrightarrow{\sim} Y_-$ as schemes over Y .

This concludes the explicit construction.

Remark 3.2. [lifting to jet-scheme]. Note that there is a one-to-one correspondence between $GL_2(\mathbb{C})$ -orbits in W and isomorphism classes of 0-dimensional torsion sheaves of length 2 on the conifold Y (i.e. the push-forward Chan-Paton sheaves on Y under associated morphisms from the Azumaya point $Space M_2(\mathbb{C})$ with the fundamental module \mathbb{C}^2) with connected support. Under this correspondence, the various special liftings-to- W in the construction above:

$$(\pi_-^{(1)'}, \pi_-^{(2)'}, \pi_-^{(3)'}, \pi_-^{(4)'}) , \quad (\pi_+^{(1)'}, \pi_+^{(3)'}) , \quad (\pi_-^{(1)'}, \pi_-^{(4)'}) ,$$

and the gluing property, up to the pointwise $GL_2(\mathbb{C})$ -action, in each tuple follow from the underlying lifting property to the related jet-schemes, which is the total space of the tangent sheaf \mathcal{T}_Y of Y in our case.

A comparison with resolutions via noncommutative desingularizations.

Consider the *conifold algebra* defined by¹³

$$\Lambda_c := \frac{\mathbb{C}\langle \xi_1, \xi_2, \xi_3 \rangle}{(\xi_1^2 \xi_2 - \xi_2 \xi_1^2, \xi_1 \xi_2^2 - \xi_2^2 \xi_1, \xi_1 \xi_3 + \xi_3 \xi_1, \xi_2 \xi_3 + \xi_3 \xi_2, \xi_3^2 - 1)},$$

where the numerator is the associative unital \mathbb{C} -algebra generated by $\{\xi_1, \xi_2, \xi_3\}$ and the denominator is the two-sided ideal generated by the elements of $\mathbb{C}\langle \xi_1, \xi_2, \xi_3 \rangle$ as indicated.

Lemma 3.3. [center of Λ_c]. ([leB-S: Lemma 5.4].) *The \mathbb{C} -algebra monomorphism*

$$\begin{array}{ccc}
\tau^\# : \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) & \longrightarrow & \Lambda_c \\
z_1 & \longmapsto & \xi_1^2 \\
z_2 & \longmapsto & \xi_2^2 \\
z_3 & \longmapsto & \frac{1}{2}(\xi_1 \xi_2 + \xi_2 \xi_1) + \frac{1}{2}(\xi_1 \xi_2 - \xi_2 \xi_1) \xi_3 \\
z_4 & \longmapsto & \frac{1}{2}(\xi_1 \xi_2 + \xi_2 \xi_1) - \frac{1}{2}(\xi_1 \xi_2 - \xi_2 \xi_1) \xi_3
\end{array}$$

realizes $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$ as the center of Λ_c .

Proposition 3.4. [representation variety of Λ_c]. ([leB-S: Proposition 5.7].) *The representation variety $Rep(\Lambda_c, M_2(\mathbb{C}))$ is a smooth affine variety with three disjoint irreducible components. Two of these components are a point. The third $Rep^0(\Lambda_c, M_2(\mathbb{C}))$ has dimension 6.*

¹³The highlight here follows [leB-S] with some change of notations for consistency and mild rephrasings to link ibidem directly with us.

This implies¹⁴ that Λ_c is a *smooth order* over $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$ and, if one defines $\text{Spec } \Lambda_c$ to be the set of two-sided prime ideals of Λ_c with the Zariski topology, then the natural morphism

$$\text{Spec } \Lambda_c \longrightarrow \text{Spec } (\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4))$$

by intersecting a two-sided prime ideal of Λ_c with the center of Λ_c gives a *smooth noncommutative desingularization* of Y . ([leB-S: Proposition 5.7].)

Up to the conjugation by an element in $GL_2(\mathbb{C})$, a \mathbb{C} -algebra homomorphism $\rho : \Lambda_c \rightarrow M_2(\mathbb{C})$ can be put into one the following three forms: (In (1) and (2) below, 0 and Id are respectively the zero matrix and the identity matrix in $M_2(\mathbb{C})$.)

$$(1) \quad \rho(\xi_1) = 0, \rho(\xi_2) = 0, \rho(\xi_3) = Id;$$

$$(2) \quad \rho(\xi_1) = 0, \rho(\xi_2) = 0, \rho(\xi_3) = -Id;$$

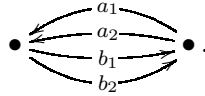
$$(3)$$

$$\rho(\xi_1) = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}, \quad \rho(\xi_2) = \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix}, \quad \rho(\xi_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Form (1) and Form (2) correspond to the two point-components in $\text{Rep}(\Lambda_c, M_2(\mathbb{C}))$ and Form (3) corresponds to elements in $\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))$. On the subvariety $\mathbb{A}_{[a_1, b_1, a_2, b_2]}^4$ of $\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))$ that parameterizes ρ of the form (3), the $GL_2(\mathbb{C})$ -action on $\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))$ reduces to the $\mathbb{C}^* \times \mathbb{C}^*$ -action

$$(a_1, b_1, a_2, b_2) \xrightarrow{(t_1, t_2)} (t_1 t_2^{-1} a_1, t_1^{-1} t_2 b_1, t_1 t_2^{-1} a_2, t_1^{-1} t_2 b_2),$$

where $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$. The pair $(\rho(\xi_1), \rho(\xi_2))$ in Form (3) realizes this $\mathbb{A}_{[a_1, b_1, a_2, b_2]}^4$ as the representation variety of the quiver



Impose the trivial $GL_2(\mathbb{C})$ -action on Y , then note that there is a natural $GL_2(\mathbb{C})$ -equivariant morphism from $\text{Rep}(\Lambda_c, M_2(\mathbb{C}))$ to Y , as the composition

$$\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \xrightarrow{\tau^\#} \Lambda_c \xrightarrow{\rho} M_2(\mathbb{C})$$

has the form

$$z_i \longmapsto 0, \quad i = 1, 2, 3, 4,$$

for ρ conjugate to Form (1) or Form (2);

$$z_1 \longmapsto a_1 b_1 Id, \quad z_2 \longmapsto a_2 b_2 Id, \quad z_3 \longmapsto a_1 b_2 Id, \quad z_4 \longmapsto a_2 b_1 Id$$

for ρ conjugate to Form (3).¹⁵ One can now follow the setting of [Ki] to define the stable structures for the $GL_2(\mathbb{C})$ -action on $\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))$. There are two different choices, θ_+ and

¹⁴Readers are referred to [leB1] for a general study of the several notions involved in this paragraph. We do not need their details here.

¹⁵Note that when restricted to $\mathbb{A}_{[a_1, b_1, a_2, b_2]}^4 \subset \text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))$, this is the morphism $\mathbb{A}_{[\xi_1, \xi_2, \xi_3, \xi_4]}^4 \rightarrow Y$ in Sec. 2 after the substitution: a_1 (here) $\rightarrow \xi_1$ (there), $a_2 \rightarrow \xi_2$, $b_1 \rightarrow \xi_3$, $b_2 \rightarrow \xi_4$.

θ_- , of such structures in the current case. The corresponding stable locus on the quiver variety $\mathbb{A}_{[a_1, b_1, a_2, b_2]}^4$ is given respectively by

$$\mathbb{A}_{[a_1, b_1, a_2, b_2]}^{4, \theta_+} = \mathbb{A}_{[a_1, b_1, a_2, b_2]}^4 - V(b_1, b_2) \quad \text{and} \quad \mathbb{A}_{[a_1, b_1, a_2, b_2]}^{4, \theta_-} = \mathbb{A}_{[a_1, b_1, a_2, b_2]}^4 - V(a_1, a_2),$$

where $V(a_1, a_2)$ (resp. $V(b_1, b_2)$) is the subvariety of $\mathbb{A}_{[a_1, b_1, a_2, b_2]}^4$ associated to the ideal (a_1, a_2) (resp. (b_1, b_2)). The corresponding GIT quotients

$$\begin{array}{ccc} \text{Rep}^0(\Lambda_c, M_2(\mathbb{C})) //^{\theta_+} GL_2(\mathbb{C}) & & \text{Rep}^0(\Lambda_c, M_2(\mathbb{C})) //^{\theta_-} GL_2(\mathbb{C}) \\ & \searrow \pi^{\theta_+} \quad \swarrow \pi^{\theta_-} & \\ & Y & \end{array}$$

recover

$$\begin{array}{ccc} Y_+ & & Y_- \\ & \searrow \pi_+ \quad \swarrow \pi_- & \\ & Y & \end{array}$$

at the beginning of the section. See [leB-S], [leB2] for the mathematical detail and [Be], [B-L], [K-W] for the SQFT/stringy origin.

From the viewpoint of the Polchinski-Grothendieck Ansatz, *both* the Azumaya-type noncommutative structure on D-branes and a noncommutative structure over Y described by $\text{Space } \Lambda_c$ come into play in the above setting. As indicated by the explicit expression for $\rho \circ \tau^\sharp$ above, any morphism $\tilde{\varphi} : \text{Space } M_2(\mathbb{C}) \rightarrow \text{Space } \Lambda_c$ has the property:

- The composition

$$\text{Space } M_2(\mathbb{C}) \xrightarrow{\tilde{\varphi}} \text{Space } \Lambda_c \xrightarrow{\tau} Y$$

is a morphism $\varphi := \tilde{\varphi} \circ \tau$ from the Azumaya point $pt^{Az} = \text{Space } M_2(\mathbb{C})$ to Y with the associated surrogate $pt_\varphi \simeq \text{Spec } \mathbb{C}$.

Thus, the new ingredient of target-space noncommutativity comes into play as another key role toward resolutions of Y in the above setting while the generalized-jet-resolution-of-singularity picture in our earlier discussion disappears.

Remark 3.5. [*world-volume noncommutativity vs. target-space(-time) noncommutativity*]. Such a “trading” between a noncommutativity target and morphisms from Azumaya schemes to a commutative target suggests a partial duality between D-brane world-volume noncommutativity and target space(-time) noncommutativity.

FIGURE 3-1.

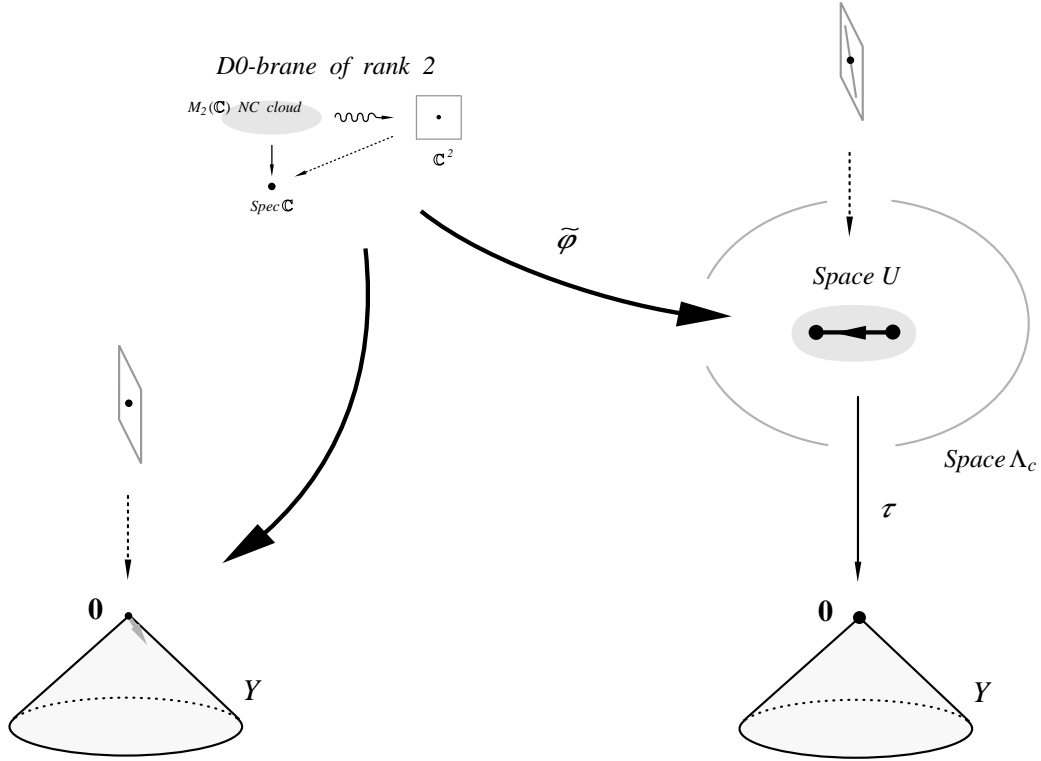


FIGURE 3-1. Trading of morphisms from $Space M_2(\mathbb{C})$ directly to the conifold Y with those to the noncommutative space $Space \Lambda_c$ over Y . Note that for generic $\rho \in Rep(\Lambda_c, M_2(\mathbb{C}))$ such that $\rho \circ \tau^\sharp = 0$, $\rho(\Lambda_c)$ is similar to the \mathbb{C} -subalgebra U of upper triangular matrices in $M_2(\mathbb{C})$. The noncommutative point $Space U$ is also smooth, with $Spec U$ consisting of two \mathbb{C} -points connected by a directed nilpotent bond. It is thus represented by a quiver $\bullet \longrightarrow \bullet$ in the figure. Furthermore, let $\tilde{\varphi} : Space M_2(\mathbb{C}) \rightarrow Space \Lambda_c$ be the corresponding morphism. Then $\tilde{\varphi}$ determines also a flag in the Chan-Paton module $\tilde{\varphi}_* \mathbb{C}^2$ on the image D0-brane $Im \tilde{\varphi}$. On the other hand, over a generic $p \neq \mathbf{0}$ on Y , the generic image of a $\tilde{\varphi}'$ that maps to p after the composition with τ will be simply $Space M_2(\mathbb{C})$.

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